XI. Of Proportion

46. When we compare two quantities, we may consider 1° either how much the one exceeds or is exceeded by the other, or 2° how often the one contains, or is contained in the other. For instance, if the numbers 4 & 12 were to be compared, I might 1° barely consider that 12 exceeds 4 by 8; or 2° I might consider that 12 contains 4 three times. From the first consideration arises the notion of Arithmetical ratio; & from the second arises the notion of Geometrical ratio or what is simply called ratio by Mathematicians; because arithmetical ratio is better expressed by the word difference. In general if there be two quanties a & b, their arithmetical ratio or difference will be expressed by a − b or b − a; & their Geo

-metrical ratio by $\frac{a}{b}$.

47. The equality or ratios is called proportion.

That is the equality of differences constitutes arithmetical proportion; & the equality of quotients constitutes geometrical proportion.

For instance if $5-3=7-5$ or if $a-b=c-d$ the numbers 5, 3, 7, 5, or the quantities a, b, c, d, are said to be in arithmetical proportion.

But if $\frac{a}{b}=\frac{c}{d}$ then are the quan=
=tities a, b, c, d, said to be in geometrical proportion; that is a, is to b, as c, is to d, the the meaning of which expression is, that a, contains or is contain'd in b, as often as c, con=
=tains, or is contained in d.

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48. Some authors use a particular notation for arith=
=metical proportion. For instance, if the
quantities, a, b, c, d, be in arithmetical proportion
they express [illegible] or thus, a. b :: c,, d. But
this is [illegible] because [illegible] $a-b=c-d$, or $b-a=d-c$
express[illegible] arithmetical proportion perfectly
well.

49. In every proportion the first & last quanti=
=ties, or terms, are called <u>extreems</u>, the others are
called <u>means</u>.

50. In every arithmetical proportion the
sum of the means is equal to the sum of the
extreems.

Thus if, a, b, c, d, be in arithmetical proportion
I say that $a+d=b+c$. For by the supposition,

$$a-b=c-d$$
$$+b$$
$$a=c-d+b$$
$$+d=\quad+d$$

$$\therefore a+d=b+c$$

Q.E.D.

51. In every Geometrical proportion the product of the ext[illegible] is equal to the product of the means.

Thus if $a : b :: c : d$, I say [illegible] $ad = bc$.

For by the suppose[illegible]

[illegible] $= \frac{bc}{d}$

$\frac{xd}{ad} =$ [illegible]

52. This property of proportional numbers is the foundation of the rule of three in arithmetick. For if, $a : b :: c : d$. Then $ad = bc$. And dividing by a we have $d = \frac{bc}{a}$.

That is if four numbers be proportional & the three first be known, the fourth may be found by multiplying the second & third together & dividing by the first; which is the common arithmetical rule.

53. The converse of the foregoing proposition is also true, that four quantities are proportional if the product of the extreems is equal to the

product of the me[illegible]

Thus if $fg = hl$ [illegible]

For

Divid. by g) $\dfrac{fg}{g} = g$ [illegible]

$f = \dfrac{hl}{g}$

dividing by h) [illegible]

$\dfrac{f}{h} = \dfrac{l}{g}$. Therefore $f : h :: l : g$. because if $\dfrac{f}{h} = \dfrac{l}{g}$

the ratio of f to [illegible] same as that of l to g, by §47.

54. Proportion is a[illegible] called analogy; & when the proportion is used simply it is always understood of Geometrical proportion. The like is to be observed of ratio, as before mention'd §46.

55. From what has been said it appears that the ratio between two quantities a & b is discovered by dividing a by b; & this quotient is called the exponent of the ratio. Thus $\dfrac{a}{b}$ is the exponent of the ratio of a to b; for the

the contra[illegible] b [illegible] is the exponent of the ratio of [illegible]

56. The fir[illegible] [illegible]tio is called the antecedent, [illegible]d the consequent.

57. A Ratio is greater as its exponent is greater: Thus the ratio of 5 to 3 is greater than the ratio of 4 to 3; because the exponent of the first ratio, or $\frac{5}{3} = 1\frac{2}{3}$, is greater than the exponent of the second, or $\frac{4}{3} = 1\frac{1}{3}$: & in general if $\frac{a}{b} > \frac{c}{d}$ the ratio of a to b is said to be greater than the ratio of c to d. For the same reason since $\frac{c}{d} < \frac{a}{b}$ the ratio of c to d is less than the ratio of a to b. This inequality of ratios is sometimes marked thus $a:b > c:d$, or $c:d < a:b$.

58. If the exponents be equal the ratios will be equal; & the quantities from the comparison of which the equal exponents arise will be proportional, according to §47. or form an Analogy (§64)

59. The ratio of two quantities is not changed by their being multiplied or divided by the same quantity.

For if $\frac{a}{b} = \frac{c}{d}$ it is also true that $\frac{am}{bm} = \frac{c}{d}$. Because if $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$ (§51) But if $ad = bc$, then $adm = bcm$ (§35) And if $adm = bcm$ then $am : bm :: c : d$. (§53) Or $\frac{am}{bm} = \frac{c}{d}$ Q.E.D.

60. A ratio is said to be the *inverse* of another when the antecedent of one is to its consequent as the consequent of the other is to its antecedent.

Thus the ratio of b to a is the inverse of the ratio of a to b. In like manner if $\frac{c}{d} = \frac{a}{b}$ the ratio of c to d will be the inverse of the ratio of b to a. i.e. in numbers, the ratio of 5 to 3 is the inverse of the ratio 3 to 5.

61. Hence a mutable, variable, or changeable quantity is said to be in the inverse ratio of another, when that is increased as this is diminished & vice versa. Thus, the time required to finish a work will be inversely as the number of laborers. Thus also the time requisite to move to a given distance will be inversely as the velocity or swiftness of the moving body. And therefore time is said to be inversely as velocity.

This consideration put arithmeticians

upon [illegible] called the inverse rule of three. But this is unnecessary: because all cases of the inverse rule may be brought to the direct rule by stating the terms proper. For if a is to b inversely as c is to q we may say directly, as [illegible] is to q, & therefore $\frac{ac}{b}=q$ as before (§ 52.)

Observe that when the antecedent of one ratio is to its consequent, as the antecedent of another ratio is to its consequent, the first quantity is said to be to the second directly as the third is to the fourth. And [illegible] quantity is said to be directly as another, when it increases or diminishes as the other increases or diminishes. Hence Mechanists say that the time a body moves is directly as the space & inversely as the velocity.

62. If various quantities be [illegible] others that are inversely as these, these may be found, by dividing unity, or the same quantity by each.

Thus $\frac{1}{a}$, $\frac{1}{b}$, are inversely as a, b; for $\frac{1}{a} : \frac{1}{b} :: b : a$. as is evident by multiplying the extremes, & means: for $\frac{1}{a} \times a = \frac{1}{b} \times b$, because each product equal to unity. In like manner the quantities $\frac{n}{a}$, $\frac{n}{b}$, are inversely as a & b.

63. The quantities $\frac{1}{a}$ & $\frac{1}{b}$, are said to be the reciprocals of a & b, & $\frac{b}{a}$, is the reciprocal of $\frac{a}{b}$. Hence reciprocals multiplied together always produce unity; for $\frac{1}{a} \times a = 1$, & $\frac{b}{a} \times \frac{a}{b} = \frac{ab}{ab} = 1$. In numbers $\frac{1}{3}$ is the reciprocal of 3 & to multiply by any quantity is the same as to divide by its reciprocal & vice versa.

64. If the first quantity is to the second as the second is to third, the three quantities are said to be in <u>continued proportion</u>. Thus a: b :: b: c, or if 3:9 :: 9:27, then are the quantities, a, b, c, or the numbers 3, 9, 27, in continued proportion. The mark of this proportion is ÷. Thus ÷ a, b, c denotes that, a:b :: b:c. And ÷ a, b, c, d, e, f, g, &c. denotes that, a:b :: b:c :: c:d :: d:e :: e:f :: f:g

65. And in this case the quantities, a, b, c, d, e, f, g, are said to be in <u>geometric progression</u>.

66. In like manner an arithmetic proportion is said to be continued when the first term differs as much from the second as the second does from the third.

67. And if there be many quantities, that the difference of any two directly following each other is always the same, those quantities are said to be in <u>arithmetic progression</u>. Thus the numbers, 3, 8, 13, are in continued arithm

proportion, & the numbers 1, 2, 3, 4, 5, or the numbers 1, 3, 5, 7, 9 &c are in arithmetic proportion: for which some use the mark ÷. Hence ÷ 6, 9, 12, 15, 18, 21, 24 denotes that these numbers are in arithmetic progression.

50. A ratio is said to be compounded of other ratios, when it is produced by multiplying the exponents of those ratios together. For instance, let a be double of b, & c triple of d; then if f is both double & triple of g; that twice triple, or thrice double or sextuple, this last ratio will be compounded of the other two, that is $\frac{f}{g} = \frac{ac}{bd}$. In like manner the ratio of 12 to 15 or of 4 to 5 is compounded of the ratio of 3 to 5 & of 4 to 3; because $\frac{12}{15} = \frac{3}{5} \times \frac{4}{3}$. So likewise the ratio of 1 to 2 is compounded of the ratios 2 to 3, & of 3 to 4: for $\frac{1}{2} = \frac{2}{3} \times \frac{3}{4} = \frac{6}{12}$. ~~[illegible]~~ also the ratio 1 to 2 is compounded of the ratios 4 to 5, 5 to 6, & 3 to 4: for $\frac{4}{5} \times \frac{5}{6} \times \frac{3}{4} = \frac{60}{120} = \frac{6}{12} = \frac{1}{2}$.

69. Hence it follows 1° that the exponent of a com=
=pounded ratio is the product of the exponents of the
ratios of which it is compounded. And 2° that any
ratios a to b, c to d, e to g, being given, quantities
in a ratio compounded of these will be found
by multiplying the antecedents together and
the consequents together. Thus the exponent of the
ratio ace to bdg, is $\frac{ace}{bdg}$ which is the product of
the three exponents $\frac{a}{b} \times \frac{c}{d} \times \frac{e}{g}$.

70. The doctrine of compound ratios is the
foundation of the compound rule of three
in arithmetic. For if any unknown quantity
q be to one known of the same kind, in a
ratio compounded of ever so many other ratios
q may easily be found. For instance, let
q be to m, in a ratio compounded of b to a &
of d to c: then $\frac{q}{m} = \frac{b}{a} \times \frac{d}{c} = \frac{bd}{ac}$. Multiply both
sides of the equation by m, & we have
$q = \frac{bdm}{ac}$.

To apply this to numbers, suppose it were required to find what quantity of [illegible] would serve 100 men for 4 months, 60 lb of meat serves 2 men for 1 month. Then [illegible] evident, that the quantity of meat will be directly in a compound ratio of the [illegible] & of the time: that is $\frac{q}{60} = \frac{100}{2} \times \frac{4}{1} = \frac{400}{2} = 200$ Therefore $q =$
$q = 60 \times 200 = 12000$. And this is clearly so; for if 60 lb serves two men 1 month, it will require 50 times 60 lb to serve 100 men for the same time, that is 3000 lb will serve 100 men for 1 month. But if 3000 is required for 1 month, it is evident, that 4 times as much [illegible] be requisite for 4 months, that is 12000 lb as before.

71. A ratio compounded of two equal ratios is called a <u>duplicate ratio</u>. Thus the ratio of

4 to 9 is duplicate of the ratio 2 to 3. For

$\frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$. So $\frac{aa}{bb}$ is duplicate of $\frac{a}{b}$. And the ratio $\frac{a}{b}$ is said to be <u>subduplicate</u> of the ratio $\frac{aa}{bb}$.

72. A <u>triplicate ratio</u> is that which is compounded of three equal ratios. Thus the ratio of $\frac{8}{27}$ is triplicate of the ratio 2 to 3. For $\frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{8}{27}$. So $\frac{a^3}{b^3}$ is triplicate of $\frac{a}{b}$. The ratio $\frac{a}{b}$ is said to be <u>subtriplicate</u> of the ratio $\frac{a^3}{b^3}$. Hence the meaning of <u>quadruplicate</u>, <u>quintuplicate</u>, <u>subquadruplicate</u> & <u>subquintuplicate</u> &c. may be understood.

73. In a continued arithmetic proportion the sum of the extremes is double of the mean. Thus in the continued arithmetic

proportion, 3, 5, 7, the sum of the extremes is $3+7$ or 10 which is double of 5 the mean term. In general any continued arithmetic proportion may be denoted thus, a, $a \pm d$, $a \pm 2d$, where the sum of the extremes will always be $2a \pm 2d$, which is double the mean, $a \pm d$

Q.E.D.

74. This mean term is called an __arithmetical mean__ between the two extremes. Hence to find an arithmetical mean between two quantities you need only take half their sum.

75. In a continued geometric proportion the product of the extremes is equal to the square of the mean term. Thus in the continued proportion, $2 : 6 :: 6 : 18$, ~~the~~ the product of the extremes 2×18, is 36, & the square of the mean 6 is also 36.

GEO. ADD. MSS. 32 (1907)

And in general if $a : b :: b : c$, then will $ac = bb$

see § 51.

76. Hence it follows, that to find a geometric mean proportional, or simply a geometric mean, or as it is often expressed a mean proportional between two quantities, you need only multiply the quantities together & extract the square root of the product. Thus if it were required to find a mean proportional between 20 & 180: their product is 3600, the square root of which ~~~~ is 60 the mean required; for $20 : 60 :: 60 : 180$.

77. The first term in a continued proportion is to the third in a duplicate ratio of the first to the second. For in the geometric proportion $a : am :: am : amm$, the ratio of a to amm, is expressed by $\frac{a}{amm} = \frac{1}{mm}$ & the ratio of a to am is expressed by $\frac{a}{am} = \frac{1}{m}$; but the ratio $\frac{1}{m^2}$ is duplicate of $\frac{1}{m}$. Because $\frac{1}{m} \times \frac{1}{m} = \frac{1}{m^2}$ Q. E. D.

78. From what has been said the analogy between ratios & fractions is evident. The exponent of a ratio may be considered as a fraction: but instead of dividing ratios into proper & improper, they may be distinguished into ratios greater & less than that of equality or of 1 to 1. Thus the ratio of 3 to 2 is greater than that of equality; but the ratio of 2 to 3 is less. The exponent of the first will be an improper fraction, & that of the second will be a proper one. And this analogy between ratios & fractions has led some authors to call the antecedant of a ratio its numerator, & the consequent its denominator. But these appellations are inconvenient in geometry; because it is certain that there are some ratios, such as

that of the diagonal of a square to its side, which cannot be expressed in numbers, and therefore strictly speaking cannot be said to have either numerator or denominator, tho these may be found to any degree required of exactness.

79. The doctrine of ratios being well understood throws great light on some parts of the theory of fractions that seem obscure to beginners. Thus the multiplication & division of fractions generally preplexes. but the difficulty vanishes when it is considered, that in every multiplication unity is to the multiplier as the multiplicand is to the product. Therefore if the multiplier be less than unity, the product must also be less than the multiplicand. For instance let $\frac{1}{2}$ be multiplied by $\frac{1}{2}$, the product according to the rule is $\frac{1}{4}$. And this is very evident;

for $1 : \frac{1}{2} :: \frac{1}{2} : \frac{1}{4}$. It being clear that if the multiplier be half of unity, the product must be one half of the multiplicand. But this is $\frac{1}{2}$, the half of which is $\frac{1}{4}$, which is therefore the true product, and the like will be evident in other cases. In division on the contrary the divisor is to unity as the dividend is to the quotient; because the quotient multiplied by the divisor must be equal to the dividend, which is the same as the product of unity into the dividend. Hence if the divisor be less than unity, the dividend will be less than the quotient. Therefore to divide for instance $\frac{1}{3}$ by $\frac{1}{2}$, the quotient by the rule will $\frac{2}{3}$, and this is very evident from the consideration of proportion. For the divisor $\frac{1}{2}$ is to 1 as the dividend

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$\frac{1}{3}$ is to the quotient. But the divisor $\frac{1}{2}$ is but half of 1, & therefore the dividend must be but half the quotient: consequently the quotient must be double the dividend, that is double of $\frac{1}{3}$; but this doubled is $\frac{2}{3}$, which is the quotient according to the rule. The same reasoning will hold good in all other cases.

20. A ratio, or fraction, may be encreased two ways; 1° either by encreasing the antecedant or numerator; or 2° by diminishing the consequent or denominator. Thus if the ratio 4 to 7, or the fraction $\frac{4}{7}$ were proposed it is evident that by adding any number, 1 for instance to the numerator 4, the fraction or the ratio of which it is the exponent will be encreased. For $\frac{4+1}{7}=\frac{5}{7}>\frac{4}{7}$. But the same may be done by diminishing the denominator. For $\frac{4}{7-1}=\frac{4}{6}>\frac{4}{7}$.

On the contrary a fraction, or the ratio of which it is the exponent, will be dimi=
=nished by diminishing its numerator or by encreasing its denominator. Thus $\frac{4-1}{7} = \frac{3}{7} < \frac{4}{7}$.
And $\frac{4}{7+1} = \frac{4}{8} < \frac{4}{7}$.

It is also to be observed that if the nume=
=rator be multiplied by any quantity or the dinominator divided by the same quantity, the fraction is encreased in the same degree precisely; & vice versa. ~~Thus~~
Thus $\frac{5\times 2}{8} = \frac{10}{8} = \frac{5}{8:2} = \frac{5}{4}$. And $\frac{5:5}{8} = \frac{1}{8} = \frac{5}{8\times 5} = \frac{5}{40}$.

It is also evident that multiplication by any fraction $\frac{b}{a}$ is the same thing as divi=
sion by its reciprocal $\frac{a}{b}$. From whence may be derived a short rule & easily remem=
bred for the division of fractions, which is

invert the ... & proceed as in multiplication

Thus $\frac{2}{3} : \frac{3}{4} = \frac{2}{3} \times \frac{4}{3} = \frac{8}{9}$. So $\frac{9}{10} : \frac{15}{16} = \frac{9}{10} \times \frac{16}{15} = \frac{144}{150} = \frac{24}{25}$.

81. When a great many ratios are proposed to be compounded into one, or when many fractions are to be multiplied together it will shorten the operation to examine what common divisors the numerators & denominators have. For by dividing any one or more numerators, and any one or more denominators by those common divisors, the product will not be altered in its value but will be expressed in smaller numbers. It being evident for instance that $\frac{a}{b} \times \frac{2b}{d} \times \frac{md}{f} = \frac{a}{1} \times \frac{2}{1} \times \frac{m}{f} = \frac{2am}{f} = \frac{2abdm}{bdf}$; but this last expression is much more complex than the former. So in numbers, if $\frac{8}{9}, \frac{9}{10}, \frac{15}{16}, \frac{8}{9}, \frac{9}{10}, \frac{8}{9}, \frac{15}{16}$, were proposed to be multiplied they might be placed & reduced thus,

$$\frac{8}{9} \times \frac{9}{10} \times \frac{15}{16} \times \frac{8}{9} \times \frac{9}{10} \times \frac{8}{9} \times \frac{15}{16} = \frac{1}{2}$$

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XII. Questions wherein the doctrine of Proportion is of use

82. I. To divide 100 into two parts which shall be to one another as 5 to 3.

$x =$ the least part.

$100 - x =$ greatest.

$x : 100 - x :: 3 : 5.$

$$\begin{array}{l} 5x = 300 - 3x \\ +3x \qquad\quad +3x \\ \hline 8x = 300 \end{array}$$

$x = \frac{300}{8} = 37\frac{1}{2}$ Therefore greatest part $= 62\frac{1}{2}$.

Or thus let the greatest $= 5x$

The least $= 3x$

$$8x = 100$$

$$x = \frac{100}{8} = 12\frac{1}{2}$$

Therefore the greatest $= 62\frac{1}{2}$.

And the least $= 37\frac{1}{2}$.

83. II. To divide 100 into five parts proportional to, 2, 3 5, 6, 8.

The le[ast] $= 2x$ Least $= 8\frac{2}{6}$

The Second $= 3x$ Second $= 12\frac{3}{6}$

Third $= 5x$ Third $= 20\frac{5}{6}$

Fourth $= 6x$ Fourth $= 25$

Fifth $= 8x$ Fifth $= 33\frac{2}{6}$

$24x = 100$. 100

$x = \frac{100}{24} = 4\frac{1}{6}$

84. III. A. has 46 guineas, & B. has 104. It is required to find what sum must be added to each mans money so that B's shall be double to A's.

$x =$ Sum to be added.

$$104 + x = \overline{46 + x} \times 2 = 92 + 2x$$
$$\quad -x \qquad\qquad\qquad -x$$
$$104 = 92 + x$$
$$-92 \qquad -92$$
$$12$$

$x = 12$.

Proof. $46 + 12 = 58$ is half of $104 + 12 = 116$.

85. IV. A merchant borrowed money at 5 p Ct interest, and 665 £. Q. what interest ought he to pay at the end of 7 months.

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The interest must follow the ratio of the sum lent & also of the time. Therefore the interest is in a ratio compounded of these; that is a number is to be found which is to 5 in a com=pound ratio of 7 months to 12 months & of 665 £. to 100 £.

x = The interest sought.

Then $x : 5 :: 7 \times 665 : 12 \times 100$.

$$x = \frac{5 \times 7 \times 665}{1200} = 19\tfrac{19}{48}.$$

86. V. Three Merchants gain'd in fellowship 1560 £. The first laid out 2000 £. & at years end drew it out again; the second put in 660 £, for 7 months; & the third 1255 £, for 5 months: How is the gain to be divided?

The gain is to be divided in the ratios of the money each brought & the time they continued it in the joint stock; therefore the problem is reduced to this, to divide 1560 into 3 parts, which shall be to each other as 2000×12, 660×7 & 1255×5, or

as, 24000, 4620, & 6275.

This problem may be solved like the 2d. question

§83, or by several rules of three. For the sum

of the numbers 24000, 4620, & 6275, or 34895

$34895 : 1560 :: 24000 : x$ = the share of the first.

Again, $34895 : 1560 :: 4620 : y$ = share of the second;

And lastly, $34895 : 1560 :: 6275 : z$ = share of the third

$= 1560 - x - y$.

87. Hence questions relating to the ~~rule~~ of fellowship in arithmetic may be solved.

88. VI. Suppose two sorts of silver, the first at 60d. oz: & the second at 68d. oz: How many ounces of each sort ought to be taken to make up a mixed mass of 50. oz. at 65d. an ounce.

Let x = number of ounces of the first sort.

y = ~~n~~ of the second sort.

Then $x + y = 50$. And $y = 50 - x$.

But as the first sort is worth 60^d an oz. the whole value of the first in the mixture will be $60x$. And the value of the second sort will be $68y$. Also 50. oz. at 65^d is $50 \times 65 = 3250^d$ the value of the mix'd mass.

Therefore $60x + 68y = 3250$

And $68y = 3250 - 60x$

$$y = \frac{3250 - 60x}{68} = 50 - x$$

Mult. $\times 68$. $3400 - 68x = 3250 - 60x$

Add $68x - 3250$. $3400 - 3250 = 68x - 60x = 8x$.

Therefore $x = \frac{150}{8} = 18\frac{3}{4}$. And $y = 50 - 18\frac{3}{4} = 31\frac{1}{4}$.

89. In like manner may other questions relating to the rule of <u>Alligation</u> in arithmetic be solved.

90. VII. The product of two numbers is 100 & their ratio as 5 to 4: What are the numbers?

Let first number $= x$

Second $= y$

Then $xy = 180$. And $y = \frac{180}{x}$.

But $x : y :: 5 : 4$. And $5y = 4x$. Or $y = \frac{4x}{5}$.

Th. $\frac{180}{x} = \frac{4x}{5}$

Mult. $\times x$) $180 = \frac{4xx}{5}$

Mult. $\times 5$) $900 = 4xx$

Div. by 4) $225 = xx$

Extr. $\surd$) $15 = x$

Ergo $y = \frac{180}{x} = 12$

XIII. Of Quadratic Equation.

91. The equation $xx = 225$ of the last prob.^{le} is as has been observ'd § 32, a quadratic equation but it is only what algebrists call a simple quadratic equation, to distinguish it from others, where the unknown quantity is of both of two & of one demension, as in the equation $xx + ax = bb$: And equations of this

kind are called <u>affected quadratic equations</u>.

92. Simple quadratic equations are easily solved by the extraction of the square root. Thus if $zz = 25$ it is evident, that $z = \sqrt{25} = 5$. Or in general if $zz = a$ the $z = \sqrt{a}$. But if $xx + 6x = 55$, the value of x is not so obvious.

93. The method in this and in the like cases is to compleat the square, which may always be done by adding to both sides of the equation, the square of half the known quantity of the second term. Thus in the Equation $xx + 6x = 55$, the known quantity of the second term $6x$ is 6, half of which is 3, & the square of 3 is 9: Adding therefore 9 to both sides of the equation, we shall have $xx + 6x + 9 = 55 + 9 = 64$. Then extracting the square root of both sides, we have $\sqrt{xx + 6x + 9} = \sqrt{64}$; or $x + 3 = 8$. Consequently

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$x = 8 - 3 = 5$. For if $x = 5$, then $xx = 25$, and $6x = 30$;

but $25 + 30 = 55$ Q.E.F.

Again let $zz + 8z = 180$

$$\begin{array}{r} zz + 8z = 180 \\ 16 \quad\ \ 16 \\ \hline zz + 8z + 16 = 196 \\ z + 4 = \sqrt{196} = 14 \\ z = 14 - 4 = 10. \end{array}$$

In general, let $xx + ax = b$.

$$\begin{array}{r} \frac{1}{4}aa = \frac{1}{4}aa \\ \hline xx + ax + \frac{1}{4}aa = \frac{1}{4}aa + b \\ x + \frac{1}{2}a = \sqrt{\frac{1}{4}aa + b} \\ x = -\frac{1}{2}a + \sqrt{\frac{1}{4}aa + b}. \end{array}$$

In like manner, if $xx - ax = b$

$$\begin{array}{r} \frac{1}{4}aa = \frac{1}{4}aa \\ \hline xx - ax + \frac{1}{4}aa = \frac{1}{4}aa + b \\ x - \frac{1}{2}a = \sqrt{\frac{1}{4}aa + b} \\ x = \frac{1}{2}a + \sqrt{\frac{1}{4}aa + b} \end{array}$$

94. If the first term, that is where x is of two dimensions have a coefficient, the equation must be divided by that coefficient; for the first term must always be pure, that is without a coefficient

Thus if $2zz+5z=133$, divide by 2, then

$zz+\frac{5}{2}z=\frac{133}{2}$. Then adding $\frac{25}{16}$.

$zz+\frac{5}{2}z+\frac{25}{16}=\frac{133}{2}+\frac{25}{16}=\frac{1064+25}{16}=\frac{1089}{16}$

Ergo $z+\frac{5}{4}=\sqrt{\frac{1089}{16}}=\frac{33}{4}$

And $z=\frac{33}{4}-\frac{5}{4}=\frac{28}{4}=7$.

Proof. $z=7$. $zz=49$. $2zz=98$.

$$\begin{array}{r} 2zz=98 \\ 5z=35 \\ \hline 2zz+5z=133 \end{array}$$

95 —— 100. Porponed to the end of Chap. 14.

XIV. Problems leading to quadratic Equations.

1 The sum of two numbers is 31 & their product 240: What are the numbers?

Let $x=$ 1st N.

Then $31-x=$ 2d N.

And $\overline{31-x}\times x=240=31x-xx$

$xx-31x=-240$

Add $\frac{961}{4}$. $xx-31x+\frac{961}{4}=\frac{961}{4}-240=\frac{1}{4}$

Then extr. ye sq. R. $x - \frac{31}{2} = \sqrt{\frac{1}{4}} = \frac{1}{2}$

$$+\frac{31}{2} \qquad = \frac{31}{2}$$

Then $x = \frac{32}{2} = 16$

And 2d. it. = ——— = 15.

To form a general rule or canon in this case:
Let the sum $= s$ & the product p.

Then $\overline{s - x} \times x = p = sx - xx$, or $xx - sx = -p$.

And $\frac{1}{4}ss$, & $xx - sx + \frac{1}{4}ss$ ~~[illegible]~~ $= \frac{1}{4}ss - p$.

And $x - \frac{1}{2}s = \sqrt{\frac{1}{4}ss - p}$.

Ergo $x = \frac{1}{2}s \pm \sqrt{\frac{1}{4}ss - p}$. This may be expressed in words thus,

1.° From the square of half the sum subtract the product.

2.° Extract the square root of the remainder.

3.° This square root added to, & subtracted from the half sum, gives the numbers required.

Ex. gr. Let $s = 12$. $p = 35$

Then 1°. $\frac{1}{2}s = 6$. $\frac{1}{4}ss = 36$ & $36 - 35 = 1$

2°. $\sqrt{1} = 1$.

3°. $6 + 1 = 7$. and $6 - 1 = 5$.

For $5 + 7 = 12$ and $5 \times 7 = 35$.

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102. II. The sum of two numbers & the sum of their squares being given to find the numbers.

Let s = sum
And q = sum of the squares
x = the 1st num.
y = 2d.

1° $x+y=s$
$y=s-x$
$yy=ss-2sx+xx$

2° $xx+yy=q$
$yy=q-xx$

$ss-2sx+xx=q-xx$
$-ss \quad +xx \quad +xx-ss$

$2)\ 2xx-2sx=q-ss$
$xx-sx=\frac{1}{2}q-\frac{1}{2}ss$
$\frac{1}{4}ss = \frac{1}{4}ss$
$xx-sx+\frac{1}{4}ss=\frac{1}{2}q-\frac{1}{4}ss.$
$x-\frac{1}{2}s=\sqrt{\frac{1}{2}q-\frac{1}{4}ss}$
$x=\frac{1}{2}s\pm\sqrt{\frac{1}{2}q-\frac{1}{4}ss}$

Ex. Let $s=12$
And $q=74$

$$\left.\begin{array}{l}\frac{1}{2}s=6\\ \frac{1}{4}ss=36\\ \frac{1}{2}q=37\end{array}\right\} x=6\pm\sqrt{37-36}=6\pm1. \text{ Ergo } x=\begin{cases}7\\5\end{cases}$$

GEO ADDL MSS 32 (1916) (35)

103. III. A person being ask'd how old he was replied, if from the square of my age you take 46 the remainder will be equal to the number of shillings in ten guineas

Let x = age

$$xx - 46 = 210$$
$$\underline{\quad 46}$$
$$xx = 256$$
$$x = \sqrt{256} = 16.$$

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104. IV. An other being ask'd the same question answered; if from the square of my age you take the number of shillings in ten guineas the remainder will be my age

Let x = age

$$xx - 210 = x$$
$$xx - x = 210$$
$$\underline{\quad \tfrac{1}{4}}$$
$$xx - x + \tfrac{1}{4} = 210\tfrac{1}{4} = \tfrac{841}{4}$$
$$x - \tfrac{1}{2} = \tfrac{29}{2} = 14\tfrac{1}{2} \text{ ergo } x = 15.$$

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105. V. A third to the same question replied; if from the square of my age you take four times my age & eleven the remainder will be the number of shillings in ten guineas, What was the age?

Let x = age

$xx-4x-11=210.$

$+11$

$xx-4x=210+11=221$

$4 \qquad\qquad 4$

$xx-4x+4=225$

$x-2=15$

Ergo $x=17.$

6. VI. To find a number to which if the square root of ten times that number be added, the sum shall be 20.

Let x = number.

$x+\sqrt{10x}=20$

$\sqrt{10x}=20-x$ square both sides.

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Then $10x = \overline{20-x}^2 = 400 - 40x + xx$

$xx - 50x = -400$

625

$xx - 50x + 625 = 625 - 400 = 225$

$x - 25 = \pm\sqrt{225} = \pm 15$

Ergo $x = 25 \pm 15 = 40$, or 10,

But 10 is the proper answer, for $10 + \sqrt{10 \times 10} =$
$= 10 + \sqrt{100} = 20$, whoever algebraically speaking
40 answers the question also: for $40 \times 10 = 400$
the root is -20 as well as $+20$ & $-20 + 40 = 20$

107. VII. The product of two numbers is 12 & the
difference of their Squares is 7; what are the
numbers?

Let the numbers be x & y

$xy = 12$ & $xx - yy = 7$

$y = \frac{12}{x}$ $\quad\quad yy = xx - 7 = \frac{144}{xx}$

$yy = \frac{144}{xx}$ $\quad\quad x^4 - 7xx = 144$

$\frac{49}{4} = 12\frac{1}{4}$

$x^4 - 7xx + \frac{49}{4} = 156\frac{1}{4}$

$xx - \frac{7}{2} = 12\frac{1}{2}$

Theref. $xx = 16$

$x = 4$

And $y = \frac{12}{4} = 3$

108. VIII. The sum of three continued proportionals is 14 & the sum of their squares is 84; to find the proportionals.

Let x = 1st

y = 2d

$\frac{yy}{x}$ = 3d.

1 $x + y + \frac{yy}{x} = 14$

2 $xx + yy + \frac{y^4}{xx} = 84$.

1st Eq. $x + \frac{yy}{x} = 14 - y$

$$xx + 2yy + \frac{y^4}{xx} = 196 - 28y + yy$$
$$-2yy \qquad\qquad\qquad -2yy$$
$$xx + \frac{y^4}{xx} = 196 - 28y - yy$$

By 2d Eq. $xx + \frac{y^4}{xx} = 84 - yy$

Th. $196 - 28y - yy = 84 - yy$

Or $196 - 28y = 84$

Or $196 - 84 = 28y = 112$

Ergo $y = 4$

But $x + y + \frac{yy}{x} = 14 = x + 4 + \frac{16}{x}$

Ergo $x + \frac{16}{x} = 14 - 4 = 10$

$xx + 16 = 10x$

$xx - 10x = -16$

25

$xx - 10x + 25 = 25 - 16 = 9$

$x - 5 = \pm 3$

$+5$

$x = 5 \pm 3$

That is $x = 8$
$x = 2$

If $x = 8$ then $8 : 4 :: 4 : 2 =$ 3d term

$x = 2$ then $2 : 4 :: 4 : 8 =$ 3d term

Ergo the three proportionals are 2, 4, 8,
or 8, 4, 2.